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Spinning gas clouds – without vorticity

B Gaffet

CEA DSM/DAPNIA/Service d'Astrophysique, CEN Saclay, 91191 Gif-sur-Yvette Cedex, France

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Abstract. Ovsiannikov and Dyson have considered an ordinary differential reduction of the gas-dynamical equations for an ideal gas which is adiabatically expanding and rotating. Gaffet has shown, based on its Painlevé property, the complete integrability of that ellipsoidal gas cloud model, when there is neither rotation nor vorticity and the gas is monatomic ($\gamma = \frac{5}{3}$), and has conjectured that the integrability might persist in more general cases including rotation. In this paper we show that the presence of vorticity in general destroys the integrability property, but the conjecture is otherwise verified, under the simplifying assumption of rotation around a fixed axis. In a future work we hope to extend the present result to Dyson's most general spinning gas cloud without vorticity.

1. Introduction

The equations of gas dynamics for an ideal gas of polytropic index γ, may be written in the form

$$\begin{aligned} &\operatorname{div}\vec{v} = -\frac{1}{(\gamma-1)}\frac{\mathrm{d}}{\mathrm{d}t}\ln T \\ &\partial_t\vec{v} = \vec{v}\Lambda\operatorname{rot}\vec{v} + T\vec{\nabla}S - \vec{\nabla}\left(\frac{\vec{v}^2}{2} + \frac{\gamma T}{\gamma-1}\right) \\ &\partial_t S + \vec{v}\cdot\vec{\nabla}S = 0 \end{aligned} \tag{1.1}$$

where $\vec{v}$ is the fluid's velocity, T is its temperature (normalized here in such a way that the specific enthalpy $H = \gamma T/(\gamma - 1)$) and $\mathrm{d}/\mathrm{d}t$ denotes the time derivative following the fluid's motion.

Ovsiannikov (1965) and Dyson (1968) have noted that the above equations are manifestly compatible with an ansatz which makes the entropy S a quadratic function and the velocity $\vec{v}$ a linear function of coordinates (with time-dependent coefficients), and the temperature T a function of time only. The gas-dynamical equations are thereby reduced to an ordinary differential system of order 18. Because the density ρ of the fluid is related to the entropy as

$$S = -\ln\rho + \frac{1}{(\gamma-1)}\ln T \tag{1.2}$$

the ansatz describes an ellipsoidally stratified fluid, with a Gaussian density profile, in a state of combined expansion and rotation.

In a Lagrangian formalism, the instantaneous configuration of the cloud is described by the matrix F which relates Eulerian and Lagrangian coordinates (x_i and α_i, respectively)

$$x_i = F_{ij}(t)\,\alpha_j \tag{1.3}$$

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and the equations governing the evolution of F assume a very simple form (Dyson 1968)

$$F_T \ddot{F} = T \tag{1.4}$$

(where the subscript T denotes transposition, and a dot represents differentiation with respect to time). Dyson has shown that they are the Hamiltonian equations of motion of a point mass in nine-dimensional Euclidean space (the space of the nine coefficients F_{ij}) in the potential $T/(\gamma - 1)$—the specific thermal energy of the fluid. Let us note that these equations clearly admit further reductions, such as those where the matrix F is restricted to being diagonal, block-diagonal or symmetric.

For a monatomic gas, with polytropic index $\gamma = \frac{5}{3}$, additional first integrals are present (Anisimov and Lysikov 1970, Gaffet 1983, 1996) which allow separability of the radial part of the point-mass motion, thereby reducing the problem to one of Hamiltonian motion on a unit hypersphere (in general, the 8-sphere). Gaffet (1996) has shown that when there is no rotation (that is, when the matrix F is diagonal), the resulting point-mass motion on the unit 2-sphere possesses the Painlevé property, a second integral (cubic in the momenta) and is a completely integrable Hamiltonian motion. In the same paper it was conjectured that the property of complete integrability might be preserved when rotation is included, i.e. when the matrix F becomes unrestricted.

In this paper, as a first step towards a fully general treatment, we considered the case of rotation of a tri-axial ellipsoid around a fixed axis $O\vec{z}$, where the matrix F assumes the block-diagonal form

$$F = \begin{pmatrix} F_{11} & F_{12} & 0 \\ F_{21} & F_{22} & 0 \\ 0 & 0 & F_{33} \end{pmatrix}. \tag{1.5}$$

The corresponding point-mass motion thus takes place in five-dimensional Euclidean space and after separation of its radial part reduces to motion on the unit 4-sphere in the potential $3T/2$. In this case the antisymmetric constant matrices of angular momentum (J_{ij}) and vorticity (K_{ij}) (Dyson 1968) have for a unique non-zero component J_{12} and K_{12}. When these constants of motion are taken into account, the problem can be reduced to one of motion on the 2-sphere, in a modified potential that includes a centrifugal potential arising from the angular momentum constant and another analogous potential arising from the vorticity constant. The Painlevé analysis yields a negative result (the equations do not possess the Painlevé property) in the seemingly simplest case where F is assumed to be symmetric (i.e. $F_{12} = F_{21}$). However, surprisingly, it is found that the Painlevé property is restored for non-symmetric F when there is no vorticity (i.e. when $K_{12} = 0$, $J_{12} \neq 0$). Owing to the existence of Dedekind's duality (Dedekind 1860; which exchanges the roles of F and F_T, of J_{ij} and K_{ij}), the Painlevé property also holds in motions with vorticity but without angular momentum (i.e. when $J_{12} = 0$, $K_{12} \neq 0$).

Let us note here that in the Eulerian formalism where the velocity field is described by a matrix A_{ij}

$$v_i = A_{ij}(t)\,x_j \tag{1.6}$$

the (hidden) higher degree of symmetry of motion without vorticity becomes manifest, as, in those cases, the matrix A is symmetric.

According to a well verified conjecture (Ablowitz and Segur 1977), the Painlevé property entails complete integrability of the corresponding motions. That is confirmed in this paper by obtaining an explicit second integral of the motion (of the sixth degree in the momenta), generalizing the integral obtained earlier in the absence of rotation.

We also study in detail the case where the new second integral I_6 vanishes, and show, using a particular example, how to perform the *separation of variables* in such cases. The resulting separable form of the differential system essentially coincides with that obtained in the diagonal case (Gaffet 1998b),

$$\begin{aligned} \ell_1'(u) &= \frac{m_1}{(\ell_1 - \ell_2)} \\ \ell_2'(u) &= -\frac{m_2}{(\ell_1 - \ell_2)} \end{aligned} \tag{1.7}$$

where $m_1 = \mu(\ell_1)$, $m_2 = \mu(\ell_2)$ and $\mu^2(\lambda)$ is a sixth-degree polynomial in λ. The independent variable u is the variable appropriate to the Painlevé expansions and is, in all cases, the well known Clebsch potential (Clebsch 1859, Seliger and Whitham 1968), sometimes called 'thermasy' (van Danzig 1939)

$$u = \int T \, \mathrm{d}t.$$

2. The reduction of F to the principal axes

2.1. The canonical decomposition of F

The most general 3×3 matrix F can be reduced (Dyson 1968) to the canonical form

$$F = O_1 D O_2$$

where D is diagonal and O_1 and O_2 are orthogonal matrices. The diagonal elements D_1, D_2, D_3 of D, are the lengths of the principal axes of the ellipsoidal cloud, while the rotation matrix O_1 determines their orientation in space, and the matrix O_2 plays an analogous role with respect to the space of Lagrangian coordinates. Dyson introduced the angular velocities A, B (or, equivalently, $\vec{\omega}$, $\vec{\varphi}$) which are the antisymmetric matrices

$$A = \begin{pmatrix} O\omega_3 - \omega_2 \\ -\omega_3 O \omega_1 \\ \omega_2 - \omega_1 O \end{pmatrix} \qquad B = \begin{pmatrix} O\varphi_3 - \varphi_2 \\ -\varphi_3 O \varphi_1 \\ \varphi_2 - \varphi_1 O \end{pmatrix} \tag{2.1}$$

defined by

$$\begin{aligned} \dot{O}_1 &= -O_1 A \\ \dot{O}_2 &= B O_2. \end{aligned} \tag{2.2}$$

The reformulation of the equations in terms of D, O_1, O_2 instead of F, presents several advantages: first, the potential energy $3T/2$ has a very simple expression in terms of D

$$T = \frac{1}{(D_1 D_2 D_3)^{2/3}} \tag{2.3}$$

(the constant numerator has been chosen as equal to unity, without loss of generality). Further, the equations of motion do not involve O_1, O_2 directly, but merely the angular velocities A and B. (In the block-diagonal case, as we shall see, the latter can be obtained explicitly without integration, using the first integrals J and K of angular momentum and vorticity). The result (Dyson's equation (35)) is a sixth-order differential system for three unknowns D_1, D_2, D_3,

$$\ddot{D}_1 + \left[(\omega_1^2 + \varphi_1^2) - (\vec{\omega}^2 + \vec{\varphi}^2)\right] D_1 + 2\omega_3\varphi_3 D_2 + 2\omega_2\varphi_2 D_3 = \frac{T}{D_1} \tag{2.4}$$

(together with the two equations that can be deduced by circular permutation of the indices), which must be completed by the differential equations governing the evolution of $\vec{\omega}$ and $\vec{\varphi}$,

$$(D_2\dot{\omega}_3 - D_1\dot{\varphi}_3) = 2(\dot{D}_1\varphi_3 - \dot{D}_2\omega_3) + D_1\varphi_1\varphi_2 + D_2\omega_1\omega_2 - 2D_3\varphi_1\omega_2 \quad (2.5a)$$

$$(D_2\dot{\varphi}_3 - D_1\dot{\omega}_3) = 2(\dot{D}_1\omega_3 - \dot{D}_2\varphi_3) + D_1\omega_1\omega_2 + D_2\varphi_1\varphi_2 - 2D_3\omega_1\varphi_2 \quad (2.5b)$$

(together with the equations deducible by circular permutation).

2.2. The angular momentum and vorticity constants

The equation of motion (1.4) immediately entails constancy of the antisymmetric matrices

$$\begin{aligned} J &\equiv F\dot{F}_T - \dot{F}F_T \\ K &\equiv F_T\dot{F} - \dot{F}_T F. \end{aligned} \quad (2.6)$$

In terms of the matrices D, A, B the above definitions become

$$\begin{aligned} J &\equiv O_1 j O_{1T} \\ K &\equiv O_{2T} k O_2 \end{aligned} \quad (2.7)$$

where

$$\begin{aligned} j &\equiv D^2A + AD^2 - 2DBD \\ k &\equiv D^2B + BD^2 - 2DAD. \end{aligned} \quad (2.8)$$

Since the latter may be viewed as 2-forms (being antisymmetric 3×3 matrices), they may equivalently be represented by their duals, $\vec{j}$ and $\vec{k}$, which are 3-vectors (in the same way that $\vec{\omega}$ and $\vec{\varphi}$ are the duals of A and B), and the duals of J and K will similarly be denoted as $\vec{J}$ and $\vec{K}$. The analogue of the rotation formula (2.7) for the duals then reads (3-vectors being treated as column vectors)

$$\begin{aligned} [\vec{J}] &= O_1[\vec{j}] \\ [\vec{K}] &= O_{2T}[\vec{k}]. \end{aligned} \quad (2.9)$$

Although $\vec{j}$ and $\vec{k}$ themselves do not, the scalars $\vec{j}^2$ and $\vec{k}^2$ obviously remain constant, and play the role of first integrals of the system (2.5) governing the evolution of A and B. In particular, in the case where F is block-diagonal (which is the subject of the next section), j_3 and k_3 being the only non-vanishing components of $\vec{j}$ and $\vec{k}$, become constants of the motion. In such cases the distinction between $\vec{j}$ and $\vec{J}$, $\vec{k}$ and $\vec{K}$, disappears altogether, and we can write

$$\begin{aligned} j_3 &= J_3 = J_{12} \\ k_3 &= K_3 = K_{12}. \end{aligned} \quad (2.10)$$

3. The block-diagonal case

When F assumes the block-diagonal form (1.5), O_1 and O_2 become matrices of rotation around the third axis, and the angular velocity matrices are accordingly restricted to their components ω_3, φ_3: the remaining components $\omega_1, \omega_2, \varphi_1, \varphi_2$ all vanish identically. In addition, as we show below, ω_3, φ_3 are then exactly calculable in terms of D_1, D_2. Consequently, the system (2.4), which now reads

$$\begin{aligned} &\ddot{D}_1 - (\omega_3^2 + \varphi_3^2)D_1 + 2\omega_3\varphi_3 D_2 = T/D_1 \\ &\ddot{D}_2 - (\omega_3^2 + \varphi_3^2)D_2 + 2\omega_3\varphi_3 D_1 = T/D_2 \\ &\ddot{D}_3 = T/D_3 \end{aligned} \quad (3.1)$$

constitutes a closed system for three unknowns only: D_1, D_2 and D_3. The differential system (2.5) determining ω_3, φ_3, becomes

$$\begin{aligned}(D_1\dot{\varphi}_3 - D_2\dot{\omega}_3) &= 2(\dot{D}_2\omega_3 - \dot{D}_1\varphi_3)\\ (D_1\dot{\omega}_3 - D_2\dot{\varphi}_3) &= 2(\dot{D}_2\varphi_3 - \dot{D}_1\omega_3)\end{aligned} \tag{3.2}$$

and is exactly integrable (without any restriction on the functions D_1, D_2) as

$$\begin{aligned}(\omega_3 + \varphi_3) &= \alpha/(D_1 - D_2)^2\\ (\omega_3 - \varphi_3) &= \beta/(D_1 + D_2)^2.\end{aligned} \tag{3.3}$$

That integrability property arises as a consequence of the laws of conservation of angular momentum (J_{12}) and vorticity (K_{12}), in terms of which the integration constants α, β are given by

$$\begin{aligned}\alpha &= J_{12} + K_{12}\\ \beta &= J_{12} - K_{12}.\end{aligned} \tag{3.4}$$

Thus the system (3.1) may be rewritten as

$$\begin{aligned}\ddot{D}_1 &= \frac{T}{D_1} + \frac{\alpha^2/2}{(D_1 - D_2)^3} + \frac{\beta^2/2}{(D_1 + D_2)^3}\\ \ddot{D}_2 &= \frac{T}{D_2} - \frac{\alpha^2/2}{(D_1 - D_2)^3} + \frac{\beta^2/2}{(D_1 + D_2)^3}\\ \ddot{D}_3 &= \frac{T}{D_3}.\end{aligned} \tag{3.5}$$

We have already noted that the present problem represents Hamiltonian motion in five-dimensional Euclidean space. The above formulation is even simpler: equations (3.5) are the equations of motion in three-dimensional Euclidean space (D_1, D_2, D_3) in a potential V_s:

$$2V_s = 3T + \frac{\alpha^2/2}{(D_1 - D_2)^2} + \frac{\beta^2/2}{(D_1 + D_2)^2}. \tag{3.6}$$

Then, by the usual time-coordinate transformation (Gaffet 1983, 1996):

$$t^* = \int \frac{\mathrm{d}t}{R^2} \tag{3.7}$$

where $R \equiv \sqrt{D_1^2 + D_2^2 + D_3^2}$ is the radial coordinate, and is a given (quadratic) function of time (Anisimov and Lysikov 1970), the problem may be further simplified to that of Hamiltonian motion on a unit 2-sphere $\left(R^2 = 1\right)$, in the same potential V_s.

It is convenient to introduce, as coordinate system over the sphere, the ratios of ellipsoid axes

$$\begin{aligned}H &\equiv D_1/D_3\\ K &\equiv D_2/D_3\end{aligned} \tag{3.8}$$

in terms of which the potential reads

$$V_s = \frac{\delta}{2}\left[\frac{3}{(HK)^{2/3}} + \frac{\alpha^2/2}{(H - K)^2} + \frac{\beta^2/2}{(H + K)^2}\right] \tag{3.9}$$

where $\delta = \left(1 + H^2 + K^2\right)$. (Note that on the sphere $R^2 = 1$ and $D_3 = 1/\sqrt{\delta}$.)

The spherical equations of motion in coordinates H, K, in an arbitrary potential $V_s(H, K)$, have been determined by Gaffet (1996) (see equation (3.29) therein) and read (now using the dot to denote derivation with respect to t^*)

$$-\frac{\mathrm{d}}{\mathrm{d}t^*}\left(\frac{\dot{H}}{\delta}\right) = \left(1+H^2\right)\frac{\partial V_s}{\partial H} + HK\frac{\partial V_s}{\partial K}$$
$$-\frac{\mathrm{d}}{\mathrm{d}t^*}\left(\frac{\dot{K}}{\delta}\right) = HK\frac{\partial V_s}{\partial H} + \left(1+K^2\right)\frac{\partial V_s}{\partial K} \tag{3.10}$$

and in the present case they take the form

$$\frac{\mathrm{d}}{\delta\,\mathrm{d}t^*}\left(\frac{\dot{H}}{\delta}\right) = \frac{\left(1-H^2\right)/H}{(HK)^{2/3}} + \frac{\alpha^2/2}{(H-K)^3} + \frac{\beta^2/2}{(H+K)^3} \tag{3.11a}$$

$$\frac{\mathrm{d}}{\delta\,\mathrm{d}t^*}\left(\frac{\dot{K}}{\delta}\right) = \frac{\left(1-K^2\right)/K}{(HK)^{2/3}} - \frac{\alpha^2/2}{(H-K)^3} + \frac{\beta^2/2}{(H+K)^3}. \tag{3.11b}$$

With regard to performing the Painlevé test, our earlier results strongly suggest introducing the new independent variable u

$$\mathrm{d}u = T\,\mathrm{d}t = \frac{\delta}{UV}\,\mathrm{d}t^* \tag{3.12}$$

and the new unknowns U and V

$$H^2 = U^3 \qquad K^2 = V^3. \tag{3.13}$$

4. The Painlevé analysis

In terms of the new variables U, V, the right-hand side of equation (3.11a), after multiplication by H, reads

$$\frac{\left(1-U^3\right)}{UV} + \frac{\left(\alpha^2+\beta^2\right)U^3\left(U^3+3V^3\right) + \left(\alpha^2-\beta^2\right)(UV)^{3/2}\left(V^3+3U^3\right)}{2\left(U^3-V^3\right)^3}.$$

Owing to the presence of a radical, $(UV)^{3/2}$, a Painlevé property of the system appears impossible unless its coefficient $\left(\alpha^2-\beta^2\right)$ vanishes. In view of the identification (see (3.4)) of $\alpha+\beta$ with $2J_{12}$, and of $\alpha-\beta$ with $2K_{12}$, that first necessary condition may be restated

$$J_{12}K_{12} = 0 \tag{4.1}$$

i.e. *the fluid motion must have either no vorticity or no angular momentum* if the system is to pass the Painlevé test. We accordingly assume $\alpha^2 = \beta^2$ from now on, and proceed with the Painlevé analysis under that restricting condition.

The equation of motion now reads

$$U''(u) = \frac{U'^2}{2U} + \frac{U'V'}{V} + \frac{2V}{3U}\left(1-U^3\right) + \frac{2\alpha^2}{3}\frac{V^2U^3\left(U^3+3V^3\right)}{\left(U^3-V^3\right)^3} \tag{4.2}$$

together with another equation obtained by exchanging the roles of U and V (without changing u and α^2).

That system admits an integral of energy, $\hat{E} = (K_s + V_s)$, where K_s is the kinetic energy (see Gaffet 1996, equation (4.3) therein)

$$
\begin{aligned}
K_s &= \frac{9}{8}\left\{ \frac{\left(1+V^3\right)U'^2}{UV^2} - 2U'V' + \frac{\left(1+U^3\right)V'^2}{U^2V} \right\} \\
V_s &= \frac{\delta}{2}\left\{ \frac{3}{UV} + \frac{\alpha^2\left(U^3+V^3\right)}{\left(U^3-V^3\right)^2} \right\}
\end{aligned}
\tag{4.3}
$$

and $\delta \equiv \left(1 + U^3 + V^3\right)$. As in our earlier work, it will be convenient to introduce the rescaling

$$m \equiv \tfrac{2}{9}\hat{E}.$$

In addition to the already manifest symmetry of the system under the exchange of U and V, it is worth noticing the following discrete symmetry (with the tilde here denoting transformed quantities)

$$
\begin{aligned}
\tilde{U} &= U e^{2i\pi/3} \\
\tilde{V} &= V e^{-2i\pi/3}
\end{aligned}
\tag{4.4}
$$

which leaves unaltered U^3, V^3 and the product UV.

The Painlevé analysis (Ince 1956, Weiss *et al* 1983) consists of an examination of the properties of the system under study in the neighbourhood of singularities of all types. The system is said to possess the Painlevé property if the only movable singularities are poles, and hence if each resonance at a movable pole is free of logarithmic singularity. Thus the Painlevé test requires a limited development of the solution near each singularity, until the point where the last resonance associated with that singularity is reached.

In the present case, singularities can occur when $U \to 0$, when $V \to 0$, when $U \to V e^{2ik\pi/3}$ and when $U \to \infty$ or $V \to \infty$. Owing to the discrete symmetry (4.4), the cases $U \to V e^{\pm 2i\pi/3}$ are not distinct from those where $(U - V) \to 0$; therefore we need only examine the three cases

(a) $U \to \infty$;
(b) $U \to 0$;
(c) $(U - V) \to 0$.

Case (a). $U \to \infty$. Then $V \to \infty$ as well. The α^2 potential term is then completely negligible and the Painlevé expansion coincides precisely with that already obtained by Gaffet (1996) (see appendix A therein). That singular branch was found to pass the Painlevé test.

Case (b). $U \to 0$. Then V tends towards some finite limit V_0. The α^2 terms are again negligible. The functions U and V turn out not to be singular at all. A Painlevé expansion may still be performed, and proceeds as in the diagonal case. (It was not done explicitly in our earlier work, since U is not properly speaking singular when $U \to 0$, but it is essentially equivalent to that of case (a), owing to the existence of a symmetry $\tilde{U} = 1/U$, which holds in that context.)

Case (c). Unlike the two preceding cases, the α^2 terms are dominant here. That branch passes the Painlevé test too. In fact, surprisingly, the symmetrical combinations UV and U^3+V^3 both satisfy the Painlevé criterion in case (c), *even in the presence of vorticity*. The breakdown of the Painlevé property in such cases $(\alpha^2 \neq \beta^2)$ paradoxically takes place in branch (b) ($U \to 0$) in spite of the fact that the α^2 and β^2 terms are then negligible up to the last resonance, through the occurrence at higher orders of terms of half-integer degree, arising from the $(\alpha^2 - \beta^2)(UV)^{3/2}$ terms in the equations of motion.

5. The second integral

5.1. Determination of the second integral

Having shown that our system passes the Painlevé test, one can safely conclude that it is completely integrable, and thus it seems likely that it should present a second integral obtainable in closed form, generalizing the integral I_2 (cubic in the momenta) obtained earlier in the diagonal case. Such a generalization, however (if restricted to being cubic) does not appear to exist. In view of the fact that I_2 appears in some fundamental formulae (Gaffet 1998a, equation (2.39) therein) only through its square, we considered the possibility that the correct generalization might be of degree six in the momenta. We further remarked that α and β have the physical meaning of momenta and therefore should not appear in the highest (sixth) degree terms in the generalized integral, which would otherwise be in effect of a degree higher than six. This means that the sixth-degree terms in the new integral (denoted I_6 hereafter) must coincide with those in I_2^2 (which are, of course, known). It is well known that from knowledge of the highest degree terms in an integral of the motion, the lower degree ones may be deduced in sequence through an overdetermined integration process that merely involves quadratures, until the degree zero is finally reached, provided, of course, that a solution does exist.

In the present case the overdetermined system is indeed compatible, and yields the second integral I_6 of the spherical motion. Before giving its expression, it will be convenient to introduce as before (see Gaffet 1998b), the 3-vector representation of the velocity variables $U'(u)$, $V'(u)$

$$\begin{aligned} \vec{\xi} &\equiv (\xi, \eta, \zeta) \\ 2\xi/\sqrt{3} &= V'/U^2 \\ 2\eta/\sqrt{3} &= -U'/V^2 \\ 2\zeta/\sqrt{3} &= (VU' - UV') \end{aligned} \tag{5.1}$$

whose components manifestly satisfy the linear constraint

$$(U^3\xi + V^3\eta + \zeta) = 0. \tag{5.2}$$

We also introduce a new variable θ

$$\theta \equiv \xi\eta - 1 + f(U, V) \tag{5.3}$$

where

$$f(U, V) \equiv \frac{(\alpha^2/3)UV}{(U^3 - V^3)^2}. \tag{5.4}$$

In terms of these the new integral I_6 admits the reasonably compact form

$$I_6 \equiv (\xi + \eta - \theta\zeta)^2 + 4f(U^3\theta + 1)(V^3\theta + 1). \tag{5.5}$$

As a consequence of the symmetries of the problem, the coordinates (U, V) merely occur through their symmetrical combinations (π, X)

$$\begin{aligned} \pi &\equiv UV \\ X &\equiv \left(U^3 + V^3\right). \end{aligned} \tag{5.6}$$

Conversely

$$\begin{aligned} 2U^3 &\equiv X + \sqrt{X^2 - 4\pi^3} \\ 2V^3 &\equiv X - \sqrt{X^2 - 4\pi^3} \end{aligned} \tag{5.7}$$

and the sum $(U + V)$ is thus given by the celebrated Cardan formulae. In terms of the new variables the expressions of f and of I_6 become

$$f \equiv \frac{\alpha^2\pi/3}{X^2 - 4\pi^3} \tag{5.8}$$

$$I_6 \equiv (\xi + \eta - \theta\zeta)^2 + 4f\left(\pi^3\theta^2 + X\theta + 1\right). \tag{5.9}$$

We note that the factor $(\xi + \eta - \theta\zeta)$ in the above expression may be rewritten as

$$(\xi + \eta - \theta\zeta) \equiv I_2 - f\zeta \tag{5.10}$$

where

$$I_2 \equiv (\xi + \eta + \zeta) - \xi\eta\zeta \tag{5.11}$$

is the second integral valid in the non-rotating case; thus I_6 manifestly reduces to I_2^2 when $\alpha^2 = 0$.

In terms of the quantities P_1, P_2, P_3

$$\begin{aligned} P_1 &\equiv \left(U^3\theta + 1\right) \\ P_2 &\equiv \left(V^3\theta + 1\right) \\ P_3 &\equiv (\theta + 1) \end{aligned} \tag{5.12}$$

we can also write

$$I_6 \equiv [(\xi + \eta + \zeta) - \zeta P_3]^2 + 4P_1P_2(P_3 - \xi\eta). \tag{5.13}$$

The existence of the second integral ensures that the Hamiltonian considered is completely integrable (Liouville integrable), and that the integration reduces to quadratures. In the following sections we will consider in more detail the case where the second integral vanishes.

5.2. *Special values of m, deduced from the Painlevé expansions*

The presence of a second integral makes it possible by fixing the values of m and I_6 to reduce the phase space to the dimension two, and the equation of motion to the second-order form

$$\begin{aligned} U'(u) &= U'(U, V) \\ V'(u) &= V'(U, V) \end{aligned} \tag{5.14}$$

where $U'(U, V)$, $V'(U, V)$ are given implicit functions. Such a Hamiltonian system is solvable by separation of variables, at least in principle. We shall assume that the separability holds *with respect to the independent variable u*, meaning that, in its separable form, the general solution of the equations of motion (5.14) (see Gaffet (1998b), equation (4.3) therein) reads

$$\begin{aligned} u &= \int \frac{\ell_1\,\mathrm{d}\ell_1}{m_1} + \int \frac{\ell_2\,\mathrm{d}\ell_2}{m_2} \\ \Phi &= \int \frac{\mathrm{d}\ell_1}{m_1} + \int \frac{\mathrm{d}\ell_2}{m_2} \end{aligned} \tag{5.15}$$

where Φ is the integration constant, ℓ_1, ℓ_2 are the new variables which make the system manifestly separable, and $m_1 = \mu(\ell_1), m_2 = \mu(\ell_2)$ are functions of ℓ_1 and ℓ_2, respectively. In differential form, that is the second-order system

$$\begin{aligned} \ell_1'(u) &= \frac{m_1}{(\ell_1 - \ell_2)} \\ \ell_2'(u) &= \frac{-m_2}{(\ell_1 - \ell_2)}. \end{aligned} \tag{5.16}$$

In simple cases the function μ may turn out to be algebraic, but its form remains unknown until the separation of variables has been completed.

Consider now a Painlevé expansion which involves three integration constants: a_0, a_1, a_2 let us say (in addition to the arbitrary constant translations of the independent variable u); when m and I_6 are kept fixed there only remains one independent integration constant, a_0 say, to which, for example, a_2 must be algebraically related:

$$F(a_0; a_2) = 0.$$

The algebraic genus of the above equation and of the equation $\mu = \mu(\ell)$ must be related. Thus the Painlevé analysis can serve to identify those special values of m and I_6 which make the function μ of lower genus. We now proceed with the determination of $F(a_0; a_2)$, choosing (arbitrarily) the singular branch for which $V \to 0$.

Its Painlevé expansion reads

$$\begin{aligned} U &= a_0\left[1 + \tfrac{2}{3}a_2u^2 + \cdots\right] \\ V &= ka_0^{1/2}u\left[1 + \tfrac{2}{3}a_1u + \cdots\right] \end{aligned} \tag{5.17}$$

where we have introduced for conciseness the symbol $k = 2\mathrm{i}/\sqrt{3}$, and a_0, a_1 and a_2 are arbitrary constants. The integral of energy is found to be the following combination of constants:

$$9m = -3a_2^2 + \frac{\left(a_0^3 + 1\right)}{a_0^3}\left[\alpha^2 - \frac{8}{k}a_1a_0^{3/2}\right] \tag{5.18}$$

and the integral I_6 may be calculated similarly. Choosing $I_6 = 0$ for simplicity selects the following value of a_1:

$$8a_1a_2a_0^{3/2} = 3k(1 - a_0^3) - 4\alpha a_2^{1/2}. \tag{5.19}$$

By combination of (5.18) and (5.19) the expected relation $F(a_0; a_2) = 0$ is obtained in the form

$$(z+1)\left[3(z-1) + \frac{4\alpha x}{k} + \alpha^2x^2\right] - 3zx^2(x^4 + 3m) = 0 \tag{5.20}$$

where $z \equiv a_0^3$ and $x \equiv a_2^{1/2}$. The corresponding curve in the (z, x)-plane has a double point (and hence a lower genus) whenever the discriminant D_{12} of that second-degree equation for z has a double root,

$$D_{12} \equiv x^{12} + 6\hat{m}x^8 - 4rx^7 + 9\hat{m}^2x^4 - 12\hat{m}rx^3 + 8r^2x^2 - 8rx + 4 \tag{5.21}$$

where $\hat{m} \equiv \left(m - \frac{1}{9}\alpha^2\right)$ and $r \equiv k\alpha/2$.

One double root occurs at $x = 1/r$ whenever

$$\hat{m} = \frac{(2r^6 - 1)}{3r^4}$$

$$m = \frac{(r^6 - 1)}{3r^4}$$

and hence

$$m = -\frac{(\alpha^6 + 27)}{9\alpha^4}. \tag{5.22}$$

This type of solution is unphysical, having a negative energy.

Another double root x is present whenever

$$\begin{aligned} m &= \tfrac{1}{3}\tau^2(2\tau^3 + 1) \\ \alpha^2 &= \frac{6}{\tau}(\tau^3 - 1)^2 \end{aligned} \tag{5.23}$$

where the parameter $\tau = \mathrm{i}x^2$. The case $\tau = 1$ ($\alpha^2 = 0, m = 1$) represents a spherically symmetric expanding cloud ($U \equiv V \equiv 1$).

For such values of m and α^2, the algebraic relation between m_1, m_2 and ℓ_1, ℓ_2, having a lower genus, must be simpler. In section 6.2 we investigate in detail the case where $\tau = \frac{1}{2}$, i.e. the case

$$I_6 = 0 \qquad m = \tfrac{5}{48} \qquad \alpha^2 = \tfrac{147}{16}.$$

6. The separation of variables

6.1. The two-dimensional phase space

In terms of the new variables the definition (4.3) of the integral of energy reads

$$3\pi m = \left(U^3\xi^2 + V^3\eta^2 + \zeta^2\right) + (1 + X)(1 + Xf) \tag{6.1}$$

where $f(U, V)$ is given by equations (5.4) and (5.8). Together with the definition (5.9) of the second integral and the linear constraint (5.2), the above equation determines a two-dimensional algebraic surface (S), the phase space, in a space of coordinates (U, V, ξ, η, ζ). It is possible to reduce the dimensionality of the embedding space to three, through the elimination of ξ and η.

First, the constant I_6 may be viewed as fixing the sum $(\xi + \eta)$:

$$\begin{aligned} (\xi + \eta - \theta\zeta) &= \Omega(\pi; X; \theta) \\ &\equiv \left\{I_6 - 4f\left(\pi^3\theta^2 + X\theta + 1\right)\right\}^{1/2}. \end{aligned} \tag{6.2}$$

Together with (5.2) that constitutes a linear system, whose solution reads

$$\xi = -\frac{\left(\zeta P_2 + V^3\Omega\right)}{\left(U^3 - V^3\right)} \qquad \eta = \frac{\left(\zeta P_1 + U^3\Omega\right)}{\left(U^3 - V^3\right)}. \tag{6.3}$$

The definition (5.3) of θ thus entails a second-degree equation for ζ,

$$\zeta^2\left(\pi^3\theta^2 + X\theta + 1\right) + \zeta\Omega\left(2\pi^3\theta + X\right) + \left\{\pi^3\Omega^2 + \left(X^2 - 4\pi^3\right)(\theta + 1 - f)\right\} = 0 \tag{6.4}$$

and another second-degree equation results from the expression (6.1) of the energy constant

$$\zeta^2\left[X\left(\pi^3\theta^2 + X + 1\right) + 4\pi^3(\theta - 1)\right] + 2\zeta\pi^3\Omega(X\theta + 2) + \left\{\pi^3 X\Omega^2 + \left(X^2 - 4\pi^3\right)[(1 + X)(1 + Xf) - 3m\pi]\right\} = 0. \tag{6.5}$$

The result of eliminating ζ between that pair of equations constitutes the equation of the surface (S), in a space of coordinates $(\pi; X; \theta)$; it is a *quartic* equation for the variable X, and one of sixth degree for θ,

$$X^2\pi^6\theta^6 + \sum_{n=1}^{5}\pi^n\theta^n\left[A_{n0} + \tfrac{1}{3}\alpha^2 A_{n1} + \tfrac{1}{9}\alpha^4 A_{n2}\right] = 0 \tag{6.6}$$

where

$$A_{50} = 6\pi X(m\pi + 1)$$
$$A_{51} = -2\pi^2(X + 2)$$
$$A_{52} = 0$$

$$A_{40} = 3\left[2mX^2 + 3\pi^2(m\pi + 1)^2\right]$$
$$A_{41} = -2\left[X(X + 1) + 3\pi^2(m\pi - 1)\right]$$
$$A_{42} = \pi^4$$

$$A_{30} = 2X^3 + 2X\left\{\left(9m^2\pi^2 + 12m\pi - 1\right) - \pi^3(4 + I_6)\right\} - 4\pi^3 I_6$$
$$A_{31} = -2\pi\left\{X(6m\pi - 1) + (3m\pi - 1) + I_6\left(\pi^3 - 4\right)\right\}$$
$$A_{32} = 2\pi^2(X + 1)$$

$$A_{20} = 3X^2\left[3m^2 + 2\pi\left(m\pi + 1 - \tfrac{1}{2}I_6\right)\right] - 3\pi X I_6(m\pi + 1) + 6\pi\left\{(m\pi + 1)\left(3m\pi - 1 - 4\pi^3\right) + \pi^3 I_6(1 - m\pi)\right\}$$
$$A_{21} = -2X^2\left(3m + \pi^2\right) - 2X\left(3m + 4\pi^2 - \tfrac{1}{2}\pi^2 I_6\right) - 2\pi^2\left\{3(m\pi + 1) + \pi^3(4 - I_6) - \tfrac{1}{2}I_6\right\}$$
$$A_{22} = (X + 1)^2$$

$$A_{10} = 6m\left\{X^3 + X\left[(3m\pi - 1) - \pi^3(4 + I_6)\right] - 2\pi^3 I_6\right\}$$
$$A_{11} = -2X^3 - 6X^2 - 2X\left[3(m\pi + 1) + \pi^3(4 - I_6)\right] - 2\left[(3m\pi + 1) + \pi^3(4 - I_6)\right]$$
$$A_{12} = 0$$

$$A_{00} = X^4 + I_6X^3 + X^2\big[3m\pi(2 - I_6) + 2(I_6 - 1) + 2\pi^3(I_6 - 4)\big]$$
$$+I_6X\big[(1 - 3m\pi) + \pi^3(I_6 - 4)\big]$$
$$+\big\{(3m\pi - 1)^2 + 2\pi^3(I_6 - 4)\big(3m\pi - 1 + \tfrac{1}{2}I_6\big) + \pi^6(I_6 - 4)^2\big\}$$
$$A_{01} = \pi(I_6 - 4)(X + 1)^2$$
$$A_{02} = 0.$$

The most noticeable thing about equation (6.6) is that the denominators $\big(U^3 - V^3\big)^2$, or $\big(X^2 - 4\pi^3\big)$ which appear in the function f, have disappeared altogether: the surface (S) remains perfectly regular in these coordinates as $\big(U^3 - V^3\big) \to 0$, in spite of the divergence of the potential terms. As a result the polynomial dependence on X does not exceed the fourth degree.

In the following we shall assume, unless otherwise stated, a vanishing second integral, $I_6 = 0$, and furthermore we choose for definiteness the case

$$m = \tfrac{5}{48} \qquad \alpha^2 = \tfrac{147}{16}$$

which satisfies the parametric representation (5.23), with the value $\tau = \frac{1}{2}$ for the parameter.

6.2. The separation of variables (case: $I_6 = 0$)

Here we illustrate the method of separating the variables, with the case where the constants are $m = \frac{5}{48}$ and $\alpha^2 = \frac{147}{16}$. The equation of the phase space (6.6) has the remarkable property of *being decomposable into a pair of second-degree equations for X*. Choosing the section by a plane $\pi = 1$ as an example, the second-degree equations read

$$\big(X^2 - SX + P\big) = 0 \tag{6.7}$$

where

$$4S = \theta\big(11 - 4\theta^2\big) + 14\sqrt{\theta + 1}$$
$$16P = \big(4\theta^2 - 49\theta - 75\big) + 56\big(\theta^2 - 2\theta - 1\big)\sqrt{\theta + 1}.$$

Whatever the value of π, the discriminant $\big(S^2 - 4P\big)$ has a double root, as predicted from the Painlevé analysis

$$\theta^2 = \frac{(4\pi - 1)}{4\pi^2} \tag{6.8}$$

and, most remarkably, its roots come in pairs with opposite signs (θ_i and $-\theta_i$). Thus in our example ($\pi = 1$), the discriminant reads

$$16\big(S^2 - 4P\big) = Z_6(\theta) - 28\sqrt{\theta + 1}\,Z_3(\theta)$$

where

$$Z_6(\theta) \equiv \big(16\theta^6 - 88\theta^4 + 105\theta^2 + 392\theta + 496\big)$$
$$Z_3(\theta) \equiv \big(4\theta^3 + 8\theta^2 - 27\theta - 8\big).$$

Its roots are given by the equation $\Delta = 0$, where

$$\Delta \equiv Z_6^2 - 784(\theta + 1)Z_3^2$$

turns out unexpectedly to be even in θ,

$$\Delta(\theta) \equiv 16\left(4\theta^2 - 3\right)^2\left(16\theta^8 - 152\theta^6 + 457\theta^4 - 3312\theta^2 + 21\,760\right). \tag{6.9}$$

An analysis of that symmetry leads to the consideration of new variables ℓ_1, ℓ_2,

$$\begin{aligned} \left(\ell_1^2 + \ell_2^2\right) &= 4\pi \\ \ell_1\ell_2 &= 2\pi\theta \end{aligned} \tag{6.10}$$

or, equivalently,

$$\begin{aligned} (\ell_1 + \ell_2) &= 2\sqrt{\pi(1+\theta)} \\ (\ell_1 - \ell_2) &= 2\sqrt{\pi(1-\theta)}. \end{aligned} \tag{6.11}$$

The solutions of the second-degree equation (6.7) for X then assume the form

$$-16X = \ell_1\ell_2\left(\ell_1^2\ell_2^2 - 11\right) + 14(\ell_1 + \ell_2) \pm m_1 m_2 \tag{6.12}$$

where

$$m_1 = \mu(\ell_1)$$

$$m_2 = \mu(\ell_2)$$

and

$$\mu^2(\lambda) \equiv (\lambda - 1)^2\nu(\lambda) \equiv \left(\lambda^6 + 11\lambda^2 - 28\lambda + 16\right)$$

$$\nu(\lambda) \equiv \left(\lambda^4 + 2\lambda^3 + 3\lambda^2 + 4\lambda + 16\right).$$

The separation of variables is thus completed, and the separable form of the equations of motion turns out to be

$$\begin{aligned} \mathrm{i}\sqrt{3}\ell_1'(u) &= \frac{-m_1}{(\ell_1 - \ell_2)} \\ \mathrm{i}\sqrt{3}\ell_2'(u) &= \frac{m_2}{(\ell_1 - \ell_2)}. \end{aligned} \tag{6.13}$$

Since $\nu(\lambda)$, having no real roots, is definite positive, the above system cannot have real solutions: in general, ℓ_1 and ℓ_2 must be complex and, if the corresponding point $(\pi; X; \theta)$ in phase space is to be real, ℓ_2 has to be either ℓ_1^* or $-\ell_1^*$.

It seems clear that there is nothing special about the case that we have considered, beyond the fact that the sixth-degree polynomial μ^2 then has a double root: it is to be expected that the separable form (6.13) still holds, though with a different form of $\mu^2(\lambda)$, whenever $I_6 = 0$.

In the present case, owing to the presence of a double root, the system (6.13) *turns out to be solvable by elliptic functions*.

Consider the elliptic function $\lambda(u)$ defined by

$$\lambda'(u) = \mathrm{i}\sqrt{\tfrac{1}{3}\nu(\lambda)}.$$

Let $\ell_1 = \lambda(u_1)$, $\ell_2 = \lambda(u_2)$, then

$$\frac{\mathrm{d}\ell_1}{\sqrt{\nu(\ell_1)}} = \frac{\mathrm{i}}{\sqrt{3}}\,\mathrm{d}u_1 \qquad \frac{\mathrm{d}\ell_2}{\sqrt{\nu(\ell_2)}} = \frac{\mathrm{i}}{\sqrt{3}}\,\mathrm{d}u_2.$$

We have, from (6.13) $\mathrm{d}u_1 + \mathrm{d}u_2 = \mathrm{d}u$ along each trajectory, i.e. $u_1 + u_2 = u$ (the integration constant is immaterial); therefore λ, ℓ_1 and ℓ_2 are algebraically related (see Abramowitz and Stegun 1972, Goursat 1949); that is to say, there exists some definite algebraic combination $\lambda(\ell_1, \ell_2)$ which is elliptic. The functions ℓ_1 and ℓ_2 themselves may involve the ζ and σ functions of the elliptic theory.

7. Singular lines and special solutions

7.1. Undeformable ellipsoids

The simplest solutions of all are those where the equivalent point mass remains motionless at a point on the unit sphere which is an extremum of the potential V_s. Then $U' = V' = 0$, and equations (4.2) reduce to a pair of algebraic equations for U and V (equivalently, π and X),

$$\begin{aligned}(X-2) &= \frac{\alpha^2 \pi X}{(X^2 - 4\pi^3)} \\ \left(X^2 - 4\pi^3\right)^2 &= \alpha^2 \pi \left(X^2 + 4\pi^3\right).\end{aligned} \tag{7.1}$$

The locus of the extrema as α^2 varies is the curve

$$4\pi^3 = X^2/(X-1). \tag{7.2}$$

Taking X as the parameter, we have

$$\begin{aligned}\alpha^2 &= \frac{(4X)^{1/3}(X-2)^2}{(X-1)^{2/3}} \\ \alpha^2 \pi &= X(X-2)^2/(X-1)\end{aligned} \tag{7.3}$$

and using

$$\begin{aligned}f(U,V) &= \frac{(X-2)}{3X} \\ \theta = (f-1) &= -\frac{2(X+1)}{3X}\end{aligned} \tag{7.4}$$

the corresponding values of the integrals of the motion are

$$\begin{aligned}m &= \frac{(X+1)^2}{9\pi} \\ \frac{27}{4} I_6 &= \frac{(X-2)^2(1-5X)}{X(X-1)}.\end{aligned} \tag{7.5}$$

The spherically symmetric expanding cloud corresponds to the value $X = 2$ of the parameter. Let us also indicate as an example the solution when $X = 3$,

$$\begin{aligned}\alpha^2 &= 1.442\,249 \\ m &= 1.709\,333 \\ I_6 &= -\tfrac{28}{81}\end{aligned} \tag{7.6}$$

and

$$\begin{aligned}U &= 1.368\,098 \\ V &= 0.760\,210\end{aligned}$$

which describes a tri-axial ellipsoidal cloud expanding and rotating, without changing shape, around one of its axes.

Finally, let us note that similar stationary solutions to the equations of motion (3.11) also exist in the presence of vorticity (i.e. when $\alpha^2 \neq \beta^2$).

7.2. The singular lines of the first type

The pair of second-degree equations (6.4) and (6.5) may be written, symbolically,

$$\begin{aligned} &A\zeta^2 + B\zeta + C = 0 \\ &a\zeta^2 + b\zeta + c = 0. \end{aligned} \tag{7.7}$$

They constitute a representation of the surface (S) (the two-dimensional phase space) embedded in the four-dimensional space (π, X, θ, ζ). Equation (6.6), which is the result of the elimination of ζ, may be written in determinant form

$$\begin{vmatrix} A & B & C & 0 \\ 0 & A & B & C \\ 0 & a & b & c \\ a & b & c & 0 \end{vmatrix} = 0. \tag{7.8}$$

A special case occurs when equations (7.7) have their two roots in common, i.e. when their coefficients are proportional.

$$\frac{A}{a} = \frac{B}{b} = \frac{C}{c}. \tag{7.9}$$

These conditions determine a locus where two sheets of the surface (S) intersect, each being associated with a different choice of ζ.

The first of the pair of conditions (7.9) determines a surface (Σ), defined (independently of I_6, m and α^2) by

$$(X + 1) = \pi^3\theta(\theta - 2). \tag{7.10}$$

On that surface (Σ) one has

$$\frac{A}{a} = \frac{B}{b} = \frac{\theta}{2\delta} \tag{7.11}$$

so that its intersection with (S) is simply given by

$$\frac{C}{c} = \frac{\theta}{2\delta}. \tag{7.12}$$

As suggested by the form of (6.6), it will be convenient to introduce a new variable ρ in place of θ,

$$\rho \equiv \pi\theta \tag{7.13}$$

then (7.12) assumes the form of a bi-quadratic equation for ρ,

$$(\pi\rho^2 - 1)(\rho^2 - 4\pi^2 + 3m) - \tfrac{1}{3}\alpha^2\pi\rho^2 = 0 \tag{7.14}$$

while (7.10) may be rewritten as

$$(X + 1) = \pi\rho(\rho - 2\pi). \tag{7.15}$$

These are the equations of a line (L_1) where the surface (S) intersects itself.

A special case occurs at the intersection (L_0) of (Σ) with the plane $\theta = 1$, where the coefficients A, B, a, b all vanish identically. On (L_0) both roots ζ become infinite, and the pair of conditions (7.9) is satisfied without the need to introduce the additional constraint (7.12).

Both lines are *generic*: they exist independently of the values of I_6, m and α^2. In the particular case studied in section 6.2, the equation of (L_0) is simply

$$(L_0) : \ell_1 = \ell_2.$$

7.3. Singular lines of the second type

Other *non-generic* singular lines on the surface (S) should appear when I_6, m or α^2 take the special values which make the curve $F(a_0; a_2) = 0$ (see section 5.2) have a double point. In particular, when $I_6 = 0$ and m and α^2 satisfy the relation (5.23), we find that there exists a singular line (L_2) represented by the equations

$$(L_2) : \begin{cases} \pi = \dfrac{(\rho^2 + \tau^2)}{2\tau} \\ (X+1) = -\frac{1}{2}(\rho^3 + 3\tau^2\rho - 2\tau^3) \end{cases}$$

where τ is the arbitrary parameter in equation (5.23). Along (L_2) equations (7.7) have only one root ζ in common, and equation (7.8) has a double root.

In the particular case where $\tau = \frac{1}{2}$, (L_2) is the locus defined by

$$\ell_1 = 1 \qquad (\text{or } \ell_2 = 1)$$

i.e. ℓ_1 is the double root of the polynomial μ^2.

7.4. The solutions of minimal energy

In the case studied in section 6.2 ($\tau = \frac{1}{2}$), the separable formulation (6.13), where $m_1^2 \equiv (\ell_1 - 1)^2 \nu(\ell_1)$, manifestly admits the exact solution

$$\ell_1(u) = 1$$

since that is a double root of m_1^2. (In contrast, the simple roots of $\nu(\ell_1)$ merely constitute extrema of $\ell_1(u)$.) That result may be extended to arbitrary values of τ, thus showing that the singular line (L_2) constitutes a *trajectory*, a particular solution to the equations of motion. When $\tau = 1$ it is a *minimal energy solution*, as $m = 1$ is the minimum possible value of the energy associated with the values $I_6 = \alpha^2 = 0$ (see Gaffet 1996, p 128). Since minimal energy solutions must be represented by singular lines (isolated real lines where the complex surface (S) intersects itself) we conclude, by continuity, that $\tau > 1$ determines the minimal possible energy $m(\alpha^2; I_6 = 0)$ (the cases $\tau < 1$, including our example $\tau = \frac{1}{2}$, being unphysical), at least in some neighbourhood of $\tau = 1$. However, as no singular trajectory of another type is encountered in the range $1 < \tau < \infty$, we conclude that this entire range corresponds precisely to the physical minimum of the energy.

8. Conclusion

The ordinary differential equation (ODE) reduction of the equations of gas dynamics proposed by Ovsiannikov (1965) and Dyson (1968), possesses special symmetries in the case of a monatomic gas (Anisimov and Lysikov 1970), which enable the radial part of the equivalent single-particle Hamiltonian motion to be separated out, thereby reducing the problem to one of motion on a unit hypersphere.

When the fluid's motion takes place without angular momentum and without vorticity, the equivalent Hamiltonian motion was found by Gaffet (1996, 1998a, b) to possess the Painlevé property and to be Liouville integrable, and it was conjectured that these properties might be preserved in more general flows including rotation. As a first step towards a fully general treatment, in this paper we have considered the case of rotation around a fixed axis, and we have proven the conjecture to be correct provided that there is no vorticity. Owing to Dedekind's duality (Dedekind 1860), the corresponding vortical flows without angular momentum are found to be completely integrable as well.

The equivalent single-particle motion can in this context be reduced to Hamiltonian motion on a unit 2-sphere, for which we obtain the second integral (denoted by I_6), which is of degree six in the momenta. For the case of a vanishing constant I_6, we indicate a general method for separating the variables, and illustrate it with an example (the case $m = \frac{5}{48}; \alpha^2 = \frac{147}{16}$). The variables ℓ_1, ℓ_2 which make the system manifestly separable, are closely related (see equations (6.10) and (6.11)) to the physical variables π and θ introduced in the text.

Finally, we obtain *the solutions of minimal energy* for the case $I_6 = 0$ (and for any fixed value of the remaining integral α^2), and show that they are expressed by elliptic functions.

These results strongly hint at the complete integrability of the spinning gas flows described by Dyson, with zero vorticity as the only constraint. We hope to be able to deal with this problem in full generality in a future work. (The new integral I_6 derived here turns out to admit a straightforward such generalization, which will be presented in a forthcoming publication.)

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